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The extended spin ladder and net models: exact dimer ground state and quantum phase transition

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Abstract. We investigate an extended spin ladder with diagonal frustrated exchanges in a wide parameter regime. By representing the model as a sum of semidefinite positive projection operators, we prove that this model has exactly a dimer ground state. Smoothly changing parameters may lead the model cover several exactly known models. Starting from this ladder model, we proposed two two-dimensional net models with exact ground states. The quantum phase transition of the ground state, due to the change of exchange strengths along perpendicular rungs, is also discussed.

PACS. 75.10.Jm Quantized spin models

1 Introduction

Over the years, there has been extensive interest on quantum spin systems with frustration. The interplay of quantum fluctuation and frustration make low-dimensional spin systems exhibit rich behaviors of magnetic properties. While many conventional techniques encounter considerable difficulty in dealing with low dimensional correlated systems, and hence the exactly solved models are especially important and play a role of touchstone. Some famous examples, such as, the Heisenberg spin 1/2 chain model solved by Bethe ansatz [1], and the Affleck-Kennedy-Lieb-Tasaki (AKLT) model with integer spin [2], provide us a solid foundation for understanding properties of one-dimensional quantum spin systems. More recently, wide interest has been concentrated on spin ladder systems [3]. Some integrable spin ladder models were proposed and solved by the Bethe ansatz method [4]. In principal, one can construct a more complicated integrable spin model basing upon the framework of quantum inverse scattering method [5], however, some unphysical spin coupling terms maybe appear to preserve the integrability [6]. Besides the Bethe ansatz method, the matrix product (MP) method can also be used to construct the spin model with exact ground state [7,8].

In general, the magnetic phases of a spin ladder system are highly related to their geometric structures. Strongly geometrical frustration allows simple dimer product to be the exact ground state of the spin system. This kind of dimerized ground state with a spin gap is especially of interest for the recently discovered [9] compound $SrCu_2(BO_3)_2$ which could be well described by the two-dimensional Shastry-Sutherland model [10,11]. Some three-dimensional models with dimerized ground states have also been investigated by several groups [12–14]. In one dimension, evidence for the singlet-dimer ground state in an $S = 1$ antiferromagnetic bond alternating chain was also reported [15]. So far, a variety of one-dimensional models with exact dimer ground states have been reported [16–22]. One typical example is the Majumdar-Ghosh (MG) model [16], whose ground state is two linearly independent products of nearest-neighbor (NN) singlets. Another example is the sawtooth (or Δ chain) model [17] whose ground state has the same form as that of the MG model. In an early publication [18], we described an asymmetric ladder model with different exchanges on legs to connect the MG model and sawtooth model smoothly. Besides these models, the spin ladder models with diagonal exchanges which have exact dimer ground states were investigated by Gelfand [20] as well as Bose and Gayen [21].

Our aim in this paper is to investigate a more general spin ladder model with different exchange integrals on both legs and diagonal lines and obtain the exact ground state of the system. Models listed above just correspond to several special cases of our generalized model and can be covered by continuously changing parameters. Furthermore, basing upon the ladder model, we extend the model to two dimension and construct several two-dimensional net models. Quantum phase transitions from dimer ground state to Haldane phase and ordered antiferromagnetic phase are also investigated.

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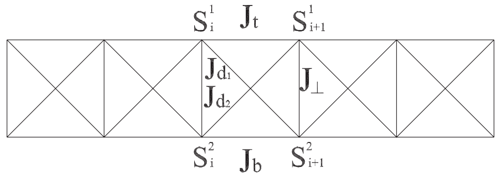

Fig. 1. The generalized spin ladder with diagonal couplings.

2 The extended spin ladder model

Now we consider a general spin ladder model

$$\begin{aligned} H = & \sum_{i=1}^{N} [J_t \ \mathbf{S}_i^1 \cdot \mathbf{S}_{i+1}^1 + J_b \ \mathbf{S}_i^2 \cdot \mathbf{S}_{i+1}^2] \\ & + \sum_{i=1}^{N} [J_{d_1} \ \mathbf{S}_i^l \cdot \mathbf{S}_{i+1}^2 + J_{d_2} \ \mathbf{S}_i^2 \cdot \mathbf{S}_{i+1}^1] \\ & + \sum_{i=1}^{N} J_\perp \ \mathbf{S}_i^1 \cdot \mathbf{S}_i^2 \end{aligned} \tag{1}$$

where the superscripts 1, 2 denote the index of the top and bottom legs and the subscript i is the site index on each leg; J_t and J_b are strengths of the NN exchanges along the top and bottom leg, respectively; J_{d_1} and J_{d_2} are strengths of the diagonal exchanges which induce frustration; and $J_\perp$ is the strength of the perpendicular exchange across rungs, as shown in Figure 1. The symmetric model with $J_t = J_b$ and $J_{d_1} = J_{d_2}$ has been investigated by Bose and Gayen [21]. We restrict our attention to the case with all strengths of the exchanges $J_t, J_b, J_{d_1}, J_{d_2} \geq 0$.

In strong coupling limit $J_\perp \to \infty$, spins on each rung would form a singlet (spin dimer). In the opposite limit $J_\perp \to -\infty$, the model corresponds to a spin-1 model. The completely dimerized state composed of a product of rung dimers

$$\Phi_D = \prod_i [1,2]_i, \tag{2}$$

is a natural choice of the ground state of the system in the limit of $J_\perp \to \infty$. Here, $[1,2]_i = ([\uparrow]_i^1 [\downarrow]_i^2 - [\downarrow]_i^1 [\uparrow]_i^2)/\sqrt{2}$ is the dimer across the ith rung. In general, when inter-rung exchanges are included, Φ_D is not an exact eigenstate of the system due to quantum fluctuation effects. However, if the constraint

$$J_t + J_b = J_{d_1} + J_{d_2}, \tag{3}$$

is concerned, the interplay of inter-rung exchanges cancels out the quantum fluctuation completely and makes Φ_D be an eigenstate. This could be easily checked by using the relation

$$\mathbf{S}_j^{1,2} \cdot (\mathbf{S}_i^1 + \mathbf{S}_i^2)[1,2]_i = 0. \tag{4}$$

It is convenient to introduce a projection operator $\mathbf{P}$ which projects a state composed of three 1/2 spins, say, $\mathbf{S}_1$, $\mathbf{S}_2$, and $\mathbf{S}_3$, into the subspace with total spin $\frac{3}{2}$,

$$\mathbf{P}[\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3] = \frac{1}{3}\left[(\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3)^2 - \frac{3}{4}\right] \cdot \tag{5}$$

Here, for convenience, we distinguish four projection operators composed of spins in four different triangles

$$\mathbf{P}_1(\mathbf{S}_i^1) = \mathbf{P}\left[\mathbf{S}_i^2, \mathbf{S}_i^1, \mathbf{S}_{i+1}^1\right], \tag{6}$$

$$\mathbf{P}_2(\mathbf{S}_i^1) = \mathbf{P}\left[\mathbf{S}_{i-1}^1, \mathbf{S}_i^1, \mathbf{S}_i^2\right], \tag{7}$$

$$\mathbf{P}_3(\mathbf{S}_i^1) = \mathbf{P}\left[\mathbf{S}_i^1, \mathbf{S}_i^2, \mathbf{S}_{i+1}^2\right], \tag{8}$$

$$\mathbf{P}_4(\mathbf{S}_i^1) = \mathbf{P}\left[\mathbf{S}_{i-1}^2, \mathbf{S}_i^2, \mathbf{S}_i^1\right]. \tag{9}$$

In terms of these projection operators, we can represent our ladder model as

$$\begin{aligned} H = \sum_{i=1}^{N} \Big\{ & \frac{3}{2}\left[J_1 \mathbf{P}_1(\mathbf{S}_i^1) + J_2 \mathbf{P}_2(\mathbf{S}_i^1) + J_3 \mathbf{P}_3(\mathbf{S}_i^1)\right. \\ & \left. + J_4 \mathbf{P}_4(\mathbf{S}_i^1)\right] - \frac{3}{4}(J_1 + J_2 + J_3 + J_4) \Big\}, \end{aligned} \tag{10}$$

where J_1, J_2, J_3 and J_4 are non-negative coupling constants. Hamiltonians (10) and (1) are exactly equivalent by enforcing

$$J_\perp = J_1 + J_2 + J_3 + J_4 \tag{11}$$

$$J_t = J_1 + J_2 \tag{12}$$

$$J_b = J_3 + J_4 \tag{13}$$

$$J_{d_1} = J_2 + J_3 \tag{14}$$

$$J_{d_2} = J_1 + J_4\,. \tag{15}$$

As long as the coefficients in equation (10) are non-negative, the Hamiltonian is a linear combination of positive semidefinite operators. Since the eigenvalue of the projector $\mathbf{P}$ is either 0 or 1, the ground state of Hamiltonian (10) is such a state that each projection operator has the lowest eigenvalue of 0 when operating on it. Therefore, a state Φ_g fulfilled $\mathbf{P}_{1,2,3,4}(\mathbf{S}_i)\Phi_g = 0$ for any i is necessarily a ground state. It can be checked that Φ_D given by (2) is such a unique state and therefore the ground state of Hamiltonian (10) with ground state energy

$$E_g = -\frac{3}{4} N J_\perp. \tag{16}$$

From equations (11–15), we can see that the generalized spin ladder model has exactly the dimer ground state in a wide restricted parameter regime. We would like to list several special cases: (i) for $J_t = J_b$ and $J_{d_1} = J_{d_1}$, the model reduces to the Bose-Gayen model [21]; (ii) if we take $J_{d_1} = 0$ or $J_{d_2} = 0$, the model is just the asymmetric ladder model with degenerate dimer ground [18,19] which interpolates the well-known M-G model [16] and the sawtooth model [17]. The M-G model and sawtooth model correspond to $J_t = J_b$ and $J_t = 0$ (or $J_b = 0$) respectively. In the case of (ii), the dimerized state Φ_D given by (2) is not the unique ground state. There is another degenerate ground state composed of the product of dimers along the diagonal direction [18,19]. A general family of models with biquadratic exchange terms has been studied in reference [23] by Kolezhuk and Mikeska, however it should be noticed that the Bose-Gayen model and our generalized model are not included in their model. For the case without biquadratic exchange terms, all the cited spin ladder models [16–21], which have a product of singlet bonds on the rungs as their exact ground states, are included in our extended model.

3 Transition to Haldane phase

It is obvious that the dimerized state Φ_D is still the ground state if $J_\perp > J_t + J_b$. For $J_\perp < J_t + J_b$, Φ_D is an eigenstate of the system, however, it may not be the ground state. It is expected that there exists such a critical value of J_c that the system has a Haldane type ground state when $J_\perp$ is smaller than J_c.

To discuss the transition from dimer phase to Haldane phase, it is instructive to represent the spin Hamiltonian in Hubbard operators defined as $X_i^{\alpha\beta} \equiv |\alpha_i\rangle\langle\beta_i|$ [4,24,25]. Here states $|\alpha_i\rangle$ span the Hilbert space of the ith rung and are given by

$$\begin{aligned} |0\rangle &= (|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle)/\sqrt{2}, \\ |1\rangle &= \ |\uparrow,\uparrow\rangle \\ |2\rangle &= (|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle)/\sqrt{2}, \\ |3\rangle &= \ |\downarrow,\downarrow\rangle\cdot \end{aligned} \tag{17}$$

The first state denotes a singlet and the latter three states compose the triplet. Using the orthogonal properties of the states, $\langle\alpha_j|\beta_j\rangle = \delta_{\alpha\beta}$, we can easily prove that $X_i^{\alpha\beta}$ obey pseudo-spin algebra, $[X_i^{\alpha\beta}, X_j^{\gamma\eta}] = [X_i^{\alpha\eta}\delta_{\beta\gamma} - X_i^{\gamma\beta}\delta_{\alpha\eta}]\delta_{ij}$. Furthermore, we can simplify the Hamiltonian through defining operator $\mathbf{T}_i$ as

$$\begin{aligned} T_i^z &= X_i^{11} - X_i^{33}, \\ T_i^+ &= \sqrt{2}(X_i^{12} + X_i^{23}), \\ T_i^- &= \sqrt{2}(X_i^{21} + X_i^{32}), \end{aligned} \tag{18}$$

which satisfy the $SU(2)$ algebra, $[T^+, T^-] = 2T^z$ and $[T^z, T^\pm] = \pm T^\pm$, and are just spin-1 operators [25]. For convenience, we would like to introduce operator $\mathbf{Y}_i$ by

$$\begin{aligned} Y_i^z &= X_i^{20} + X_i^{02}, \\ Y_i^+ &= \sqrt{2}(X_i^{03} - X_i^{10}), \\ Y_i^- &= \sqrt{2}(X_i^{30} - X_i^{01}) \end{aligned} \tag{19}$$

which do not satisfy the $SU(2)$ algebra and are introduced only for writing simplification. With these notations, equation (1) with constraint (3) can be represented as

$$H = \sum_{i=1}^{N} \left(-J_\perp X_i^{00} + J'\mathbf{T}_i \cdot \mathbf{T}_{i+1}\right) \tag{20}$$

$$+ \sum_{i=1}^{N} \delta_1 \left(\mathbf{T}_i \cdot \mathbf{Y}_{i+1} + \mathbf{Y}_i \cdot \mathbf{T}_{i+1}\right) \tag{21}$$

$$+ \sum_{i=1}^{N} \delta_2 \left(\mathbf{T}_i \cdot \mathbf{Y}_{i+1} - \mathbf{Y}_i \cdot \mathbf{T}_{i+1}\right) \tag{22}$$

where $J' = (J_t + J_b)/2$, $\delta_1 = (J_t - J_b)/2$ and $\delta_2 = (J_{d_2} - J_{d_1})/2$. Here, we have shifted the above Hamiltonian a constant $JN/4$ in comparison with equation (10). In the form of Hubbard operators, it is clear that operators X_i^{00} and $\mathbf{T}_i$ are nonzero only when they are operating on the singlet and triplet state of the ith rung respectively, however, operators $\mathbf{Y}_i$ make the translation take place between the singlet and triplet. In the symmetrical case, where $J_t = J_b = J_{d_1} = J_{d_2}$ [21,25], no terms including $\mathbf{Y_i}$ exist, the ground state of the system is either a dimer phase or a Haldane phase [25] depending on the value of $J_\perp/J'$.

One can analytically estimate the ground state energy of the Haldane phase by taking the trial ground wavefunction $|\psi^0\rangle$ as the type of MP variational wavefunction [7] given by

$$|\psi^0\rangle = \mathrm{Tr}\left(\prod_{j=1}^{L} \otimes g_j\right), \tag{23}$$

where g_j has the following matrix form

$$g_j = \begin{pmatrix} a|2_j\rangle & -a\sqrt{2}|1_j\rangle \\ a\sqrt{2}|3_j\rangle & -a|2_j\rangle \end{pmatrix} \tag{24}$$

with $a = 1/\sqrt{3}$. The wavefunction is rotationally invariant in the spin 1 space and is the exact ground state of the AKLT model [2]. Following the standard method of Klümper *et al.*, one can get the variational ground energy of $H_0 := H|_{\delta_{1,2}=0}$ basing on the MP variational wavefunction, which is

$$E_g = -\frac{4}{3}J'. \tag{25}$$

This result coincides well with the numerical result of $E_g = -1.401J'$ by the density matrix renormalization group (DMRG) method [26]. The ground state of the symmetric ladder model is therefore products of either singlets or triplets with ground energy [25]

$$\frac{E_g}{N} = \begin{cases} -J_\perp & \text{if } J_\perp > J_c \\ -J_c & \text{if } J_\perp < J_c \end{cases} \tag{26}$$

where the transition exchange J_c, which takes $4/3J'$ for the variational result or $1.401J'$ for the result of DMRG, is determined by the ground energy of spin-1 chain.

For the asymmetric ladder with $J_t \neq J_b$ or $J_{d_1} \neq J_{d_2}$, we must consider the term, $H_1 = \sum_{i=1}^{N}(\delta_1 h_{i,i+1}^{\delta_1} + \delta_2 h_{i,i+1}^{\delta_2})$ with $h_{i,i+1}^{\delta_1} = \mathbf{T}_i \cdot \mathbf{Y}_{i+1} + \mathbf{Y}_i \cdot \mathbf{T}_{i+1}$ and $h_{i,i+1}^{\delta_2} = \mathbf{T}_i \cdot \mathbf{Y}_{i+1} - \mathbf{Y}_i \cdot \mathbf{T}_{i+1}$. For small δ_1 and δ_2, we can treat H_1 as a perturbation to H_0. It is obvious that H_1 gives zero when acting on the singlet ground state. However, it will transform some triplets to singlets when applying it on the variational ground wavefunction (23). We can calculate its correction to the ground state energy by the perturbation method. Firstly we examine the first order perturbation

$$E_1 = \langle\psi^0|H_1|\psi^0\rangle$$
$$= \delta_1 \sum_{i=1}^{N}\langle\psi^0|\psi_i^1(\delta_1)\rangle + \delta_2 \sum_{i=1}^{N}\langle\psi^0|\psi_i^1(\delta_2)\rangle, \quad (27)$$

where $|\psi_i^1(\delta_{1,2})\rangle = h_{i,i+1}^{\delta_{1,2}}|\psi^0\rangle$ can be explicitly expressed as

$$|\psi_i^1(\delta_{1,2})\rangle = \mathrm{Tr}\left(\prod_{j=1}^{i}\otimes g_j \; M_{i,i+1}^{\delta_{1,2}} \prod_{k=i+1}^{N}\otimes g_k\right), \quad (28)$$

with

$$M_{i,i+1}^{\delta_{1,2}} = h_{i,i+1}^{\delta_{1,2}}(g_i \otimes g_{i+1}). \quad (29)$$

It follows that the first order perturbation gives zero, $E_1 = 0$. This is consistent with the analysis directly from the symmetry consideration. The physical properties of the ladder model (1) should not be affected by exchanging J_t and J_b (J_{d_1} and J_{d_2}), therefore only the even order corrections are available. The second order perturbation E_2 is calculated *via*

$$E_2 = -\left\langle \psi^0 \left| H_1 \frac{1}{H_0 - E_0} H_1 \right| \psi^0 \right\rangle \cdot \quad (30)$$

After some straightforward calculation, we have

$$E_2/N = \frac{1}{2}({\delta_1}^2 + {\delta_2}^2). \quad (31)$$

4 Spin net model

Now we can easily generalize the spin ladder model to a net model. We consider a double-layer model defined in two dimensions, where each layer has $N \times M$ sites and couples to another layer through inter-layer exchanges J_{d_1}, J_{d_2} and $J_\perp$. The intra-layer exchanges J_t and J_b on each layer may be different. The Hamiltonian of the net model shown in Figure 2 is given by

$$H_{net} = \sum_{i,j=1}^{N,M} J_t(\mathbf{S}_{i,j}^1 \cdot \mathbf{S}_{i,j+1}^1 + \mathbf{S}_{i,j}^1 \cdot \mathbf{S}_{i+1,j}^1)$$
$$+ \sum_{i,j=1}^{N,M} J_b(\mathbf{S}_{i,j}^2 \cdot \mathbf{S}_{i,j+1}^2 + \mathbf{S}_{i,j}^2 \cdot \mathbf{S}_{i+1,j}^2)$$
$$+ \sum_{i,j=1}^{N,M} J_{d_1}\left(\mathbf{S}_{i,j}^1 \cdot \mathbf{S}_{i,j+1}^2 + \mathbf{S}_{i,j}^1 \cdot \mathbf{S}_{i+1,j}^2\right)$$
$$+ \sum_{i,j=1}^{N,M} J_{d_2}\left(\mathbf{S}_{i,j}^2 \cdot \mathbf{S}_{i,j+1}^1 + \mathbf{S}_{i,j}^2 \cdot \mathbf{S}_{i+1,j}^1\right)$$
$$+ \sum_{i,j=1}^{N,M} J_\perp \mathbf{S}_{i,j}^1 \cdot \mathbf{S}_{i,j}^2 \; . \quad (32)$$

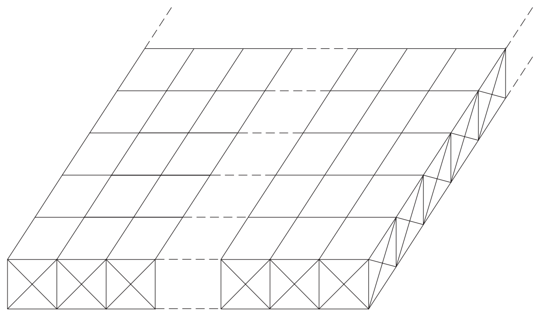

Fig. 2. The generalized spin net model.

Here the superscripts $\alpha = 1, 2$ denote labels of the top and bottom layers. $J_\perp$ is the perpendicular inter-layer exchange and $J_{d_{1,2}}$ are the diagonal inter-layer exchanges. Periodic boundary is assumed, which is equivalent to taking $M+1 = 1$ and $N+1 = 1$. A special case of this double-layer model has been investigated in reference [27], where the net model is a direct generalization of the Bose-Gayren ladder model.

It is clear that every slice of the double-layer net is just a ladder which has the same form of Hamiltonian (1). Thus, we find that the ground state of the net model is given by products of all perpendicular singlet pairs

$$\Phi_D = \prod_{i,j=1}^{M,N} \frac{1}{\sqrt{2}}\left([\uparrow]_{i,j}^1[\downarrow]_{i,j}^2 - [\downarrow]_{i,j}^1[\uparrow]_{i,j}^2\right) \quad (33)$$

when condition (3) and

$$J_\perp = 2(J_t + J_b) \quad (34)$$

are satisfied. We notice that the value of $J_\perp$ is double of that of the spin ladder, which comes from the case that two ladders cross at the perpendicular bond. More rigorous proof can be made directly from representing the net model as a sum of projection operators as in the spin ladder case. Certainly, the product of singlet is still the ground state for $J_\perp > 2(J_t + J_b)$.

Furthermore, we can even generalize the net model to the case including next-next-nearest-neighboring (NNNN) couplings. As shown in Figure 3, the Hamiltonian including intra-layer diagonal exchanges and inter-layer NNNN diagonal exchanges can be written as

$$H = \sum_{i,j=1}^{N,M} J_t'(\mathbf{S}_{i,j}^1 \cdot \mathbf{S}_{i+1,j+1}^1 + \mathbf{S}_{i,j}^1 \cdot \mathbf{S}_{i-1,j+1}^1)$$
$$+ \sum_{i,j=1}^{N,M} J_b'(\mathbf{S}_{i,j}^2 \cdot \mathbf{S}_{i+1,j+1}^2 + \mathbf{S}_{i,j}^2 \cdot \mathbf{S}_{i-1,j+1}^2)$$
$$+ \sum_{i,j=1}^{N,M} J_{d_1}'\left(\mathbf{S}_{i,j}^1 \cdot \mathbf{S}_{i+1,j+1}^2 + \mathbf{S}_{i,j}^1 \cdot \mathbf{S}_{i-1,j+1}^2\right)$$
$$+ \sum_{i,j=1}^{N,M} J_{d_2}'\left(\mathbf{S}_{i,j}^2 \cdot \mathbf{S}_{i+1,j+1}^1 + \mathbf{S}_{i,j}^2 \cdot \mathbf{S}_{i-1,j+1}^1\right)$$
$$+ H_{net}, \quad (35)$$

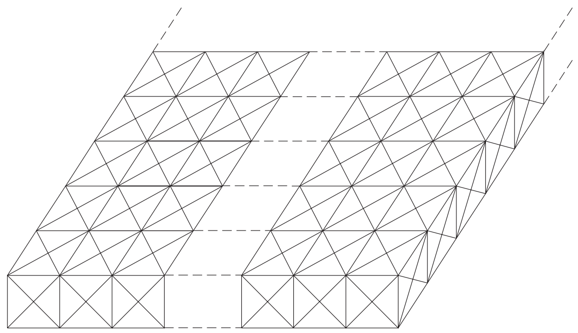

Fig. 3. The spin net model corresponding to equation (35).

where H_{net} is the Hamiltonian given by equation (32), J'_t, J'_b, J'_{d_1} and J'_{d_2} denote the top and bottom intra-layer diagonal exchanges and the inter-layer NNNN diagonal exchanges respectively.

This Hamiltonian can also be represented as a sum of positive semidefinite projected operators as long as equation (3) and the constraints

$$J'_t + J'_b = J'_{d_1} + J'_{d_2}, \tag{36}$$

$$J_\perp = 2(J_t + J_b + J'_t + J'_b) \tag{37}$$

are satisfied. Thus it is easy to prove that its ground state is exactly given by equation (33).

Similar to the spin ladder model, a quantum phase transition is expected to appear when the strength of the vertical exchange $J_\perp$ goes down to a critical value. This is more clear by representing the spin net model in Hubbard operators as

$$H_{net} = \sum_{i,j=1}^{N,M} \left[-J_\perp X_{i,j}^{00} + J' \mathbf{T}_{i,j} \cdot (\mathbf{T}_{i+1,j} + \mathbf{T}_{i,j+1}) \right] \tag{38}$$

$$+ \sum_{i,j=1}^{N,M} [(\delta_1 + \delta_2)\mathbf{T}_{i,j} \cdot (\mathbf{Y}_{i+1,j} + \mathbf{Y}_{i,j+1}) + (\delta_1 - \delta_2)\mathbf{Y}_{i,j} \cdot (\mathbf{T}_{i-1,j} + \mathbf{T}_{i,j-1})] \tag{39}$$

where $J_\perp = 2(J_t + J_b)$, J' and $\delta_{1,2}$ are same with those in the ladder model. In the symmetric case with $\delta_{1,2} = 0$, there are no terms of transforming singlet and triplet. Therefore, when $J_\perp > J_c$, the ground state of the spin net model is completely a dimerized ground state. Here, J_c is determined by the ground state of a square-lattice spin-1 model [28]. Comparing with the known numerical result [28], we have $J_c = 2.332J'$. For $J_\perp < J_c$ the spin net model, which is equivalent to the square-lattice spin-1 model, has an ordered ground state with long-range Néel order and the spin gap also vanishes. Therefore, a disorder-order phase transition appears with decreasing the strength of the exchange $J_\perp$. Like the spin ladder model, this transition is also a first-order transition.

For the asymmetric model we need consider $\delta_{1,2}$ term. When the system is in the dimerized ground state, $\delta_{1,2}$ term has no effect on the ground state, however, it brings correction to the ordered ground state with the Néel order. For small $\delta_{1,2}$ we find that this term will not destroy the long range order and it can be taken as a perturbation of the spin-1 model.

5 Conclusion

In summary, we investigate a generalized spin ladder model with NN and NNN spin couplings. By representing the spin ladder model as a sum of a series of projection operators, we exactly prove that the generalized ladder model has the dimer ground state in a wide parameter region. Furthermore, we generalize the model to two dimension and propose two double-layer models with exact dimer ground states. We also study the quantum phase transition induced by changing the vertical coupling parameter $J_\perp$.

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References

1. H.A. Bethe, Z. Phys. **71**, 205 (1931)
2. I. Affleck, T. Kennedy, E. Lieb, H. Tasaki, Phys. Rev. Lett. **59**, 799 (1987).
3. E. Dagotto, T.M. Rice, Science, **271**, 618 (1996)
4. Y. Wang, Phys. Rev. B. **60**, 9236 (1999)
5. V.E. Korepin, N.M. Bogoliubov, A.G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, 1997)
6. S. Chen, Y. Wang, F.C. Pu, J. Phys. A. **31**, 4619 (1998)
7. A. Klümper, A. Schadschneider, J. Zittartz, Europhys. Lett. **24**, 293 (1993)
8. A.K. Kolezhuk, H.J. Mikeska, Phys. Rev. Lett. **80**, 2709 (1998)
9. H. Kageyama *et al.*, Phys. Rev. Lett. **82**, 3168 (1999)
10. S. Miyahara, K. Ueda, Phys. Rev. Lett. **82**, 3701 (1999)
11. B.S. Shastry, B. Sutherland, Physica B **108**, 1069 (1981)
12. K. Ueda, S. Miyahara, J. Phys. Cond. Matt. **11**, L175 (1999)
13. N. Surendran, R. Shankar, Phys. Rev. B. **66**, 024415 (2002)
14. S. Chen, H. Büttner, Eur. Phys. J. B **29**, 15 (2002)
15. Y. Narumi *et al.*, Phys. Rev. Lett. **86**, 324 (2001)
16. C.K. Majumdar, D.K. Ghosh, J. Math. Phys. **10**, 1388 (1969); **10**, 1399 (1969); **26**, 5257 (1982)
17. D. Sen, B.S. Shastry, R.E. Walstedt, R. Cava, Phys. Rev. B. **53**, 6401 (1996); T. Nakamura, K. Kubo, Phys. Rev. B. **53**, 6393 (1996)
18. S. Chen, H. Büttner, J. Voit, Phys. Rev. Lett. **87**, 087205 (2001)
19. S. Sarkar, D. Sen, Phys. Rev. B. **65**, 172408 (2002)
20. M.P. Gelfand, Phys. Rev. B. **43**, 8644 (1991)
21. I. Bose, S. Gayen, Phys. Rev. B. **48**, 10653 (1993)
22. J. Richter, N.B. Ivanov, J. Schulenburg, J. Phys. Cond. Matt. **10**, 3635 (1998)
23. A.K. Kolezhuk, H.J. Mikeska, Int. J. Mod. Phys. B. **12**, 2325 (1998)
24. J.B. Parkinson, J. Phys. C. **12**, 2873 (1979)
25. Y. Xian, Phys. Rev. B. **52**, 12485 (1995)
26. S.R. White, D.A. Huse, Phys. Rev. B. **48**, 3844 (1993)
27. H.Q. Lin, J.L. Shen, J. Phys. Soc. Jpn **69**, 878 (1998)
28. H.Q. Lin, V.J. Emery, Phys. Rev. B. **40**, 2730 (1989)